PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 2000 **358**, 21-39
doi: 10.1098/rsta.2000.0517

THE ROYAL SOCIETY

A unified mathematical language for physics and engineering in the 21st century

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The late 18th and 19th centuries were times of great mathematical progress. Many new mathematical systems and languages were introduced by some of the millennium's greatest mathematicians. Amongst these were the algebras of Clifford and Grassmann. While these algebras caused considerable interest at the time, they were largely abandoned with the introduction of what people saw as a more straightforward and more generally applicable algebra: the *vector algebra* of Gibbs. This was effectively the end of the search for a unifying mathematical language and the beginning of a proliferation of novel algebraic systems, created as and when they were needed; for example, spinor algebra, matrix and tensor algebra, differential forms, etc.

In this paper we will chart the resurgence of the algebras of Clifford and Grassmann in the form of a framework known as *geometric algebra* (GA). Geometric algebra was pioneered in the mid-1960s by the American physicist and mathematician, David Hestenes. It has taken the best part of 40 years but there are signs that his claim that GA is the universal language for physics and mathematics is now beginning to take a very real form. Throughout the world there are an increasing number of groups who apply GA to a range of problems from many scientific fields. While providing an immensely powerful mathematical framework in which the most advanced concepts of quantum mechanics, relativity, electromagnetism, etc., can be expressed, it is claimed that GA is also simple enough to be taught to schoolchildren! In this paper we will review the development and recent progress of GA and discuss whether it is indeed the unifying language for the physics and mathematics of the 21st century. The examples we will use for illustration will be taken from a number of areas of physics and engineering.

Keywords: geometric/Clifford algebra; geometry; quantum mechanics; relativity; gravity; computer vision; buckling

1. Introduction

Today, high school students studying for A levels, or their equivalent, in the sciences will be introduced to the concept of *vectors*—directed line segments—and taught how to manipulate vectors using classical *vector algebra*. This is effectively the algebra introduced by Gibbs towards the end of the 19th century; it has changed little

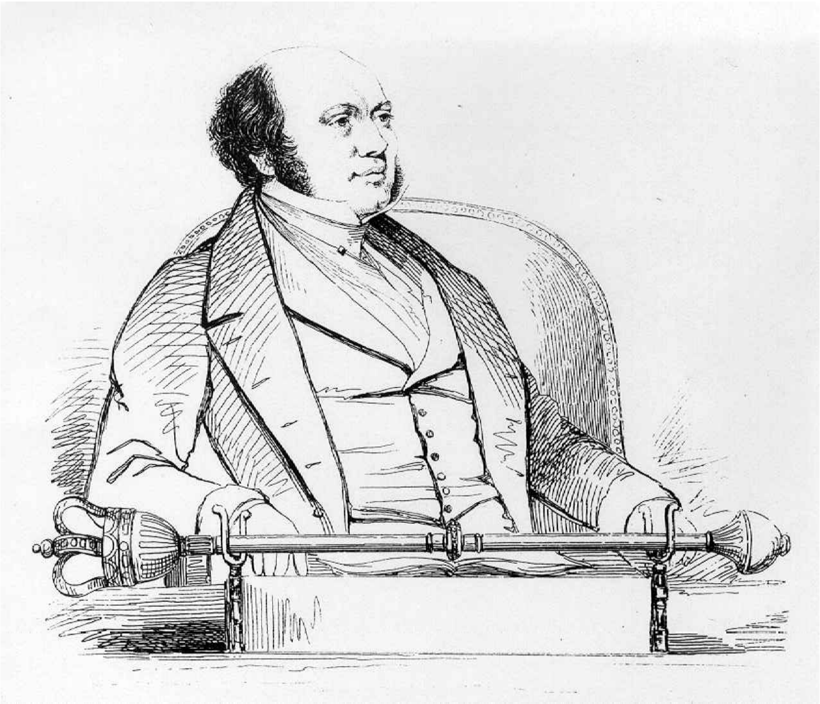

Figure 1. William Rowan Hamilton 1805–1865. Inventor of quaternions, and one of the key scientific figures of the 19th century.

since then. Those students become practised in the art of vector algebra and see how successful it is in expressing much of two- and three-dimensional geometry. Manipulation of the system becomes almost second nature. One can see how hard it then is to abandon this familiar, and apparently successful, system in favour of a new algebra (geometric algebra (GA)) that has additional rules and unconventional concepts. However, for a moderate investment of time and effort put into learning GA, the reward is to have at one's disposal a tool that allows the user to penetrate into even the most high-powered areas of current scientific research. As we move into the 21st century, we have reached the stage where to do research in the physical sciences is often to specialize in one, usually *very limited*, area. However, it has always been the case that great advantages are to be gained from interactions between fields, something that is becoming increasingly difficult but increasingly desirable. We envisage that the new millennium will see the push for interdisciplinary activity increase manyfold. In the following sections, we attempt to give the reader some evidence that GA may be the best hope we currently have of attaining the goal of a unifying mathematical language for modern science.

2. Some history

A problem that occupied many eminent mathematicians of the early 19th century was how best to represent rotations mathematically in three dimensions, i.e. ordinary space. Hamilton (see figure 1) spent much of his later life working on this problem

Figure 2. Hermann Gunther Grassmann (1809–1877). German mathematician and schoolteacher, famous for the algebra that now bears his name.

and eventually produced the *quaternions*, which were a generalization of the complex numbers (see later) to three dimensions (Hamilton 1844). The algebra contains four elements,

$$\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\},$$

which satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1.$$

While the elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are often referred to as vectors, we shall see later that they do not have the properties of vectors. Despite the clear utility of the quaternions, there was always a slight mystery and confusion over their nature and use. Today, quaternions are still used to represent three-dimensional rotations in many fields since it is recognized that they are a very efficient way of carrying out such operations. However, the confusion still persists, and a deep and detailed understanding of the quaternions has been lost to a generation.

While Hamilton was developing his quaternionic algebra, Grassmann (see figure 2) was formulating his own algebra (Grassman 1844, 1877), the key to which was the introduction of the *exterior* or *outer product*; we denote this outer product by $\wedge$, so that the outer product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a} \wedge \mathbf{b}$. This product has certain features. One such feature is its *associativity*, i.e.

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}.$$

This tells us that the way in which we group the terms together in the outer product does not matter. The other feature is *anticommutativity*, that is, if we reverse the

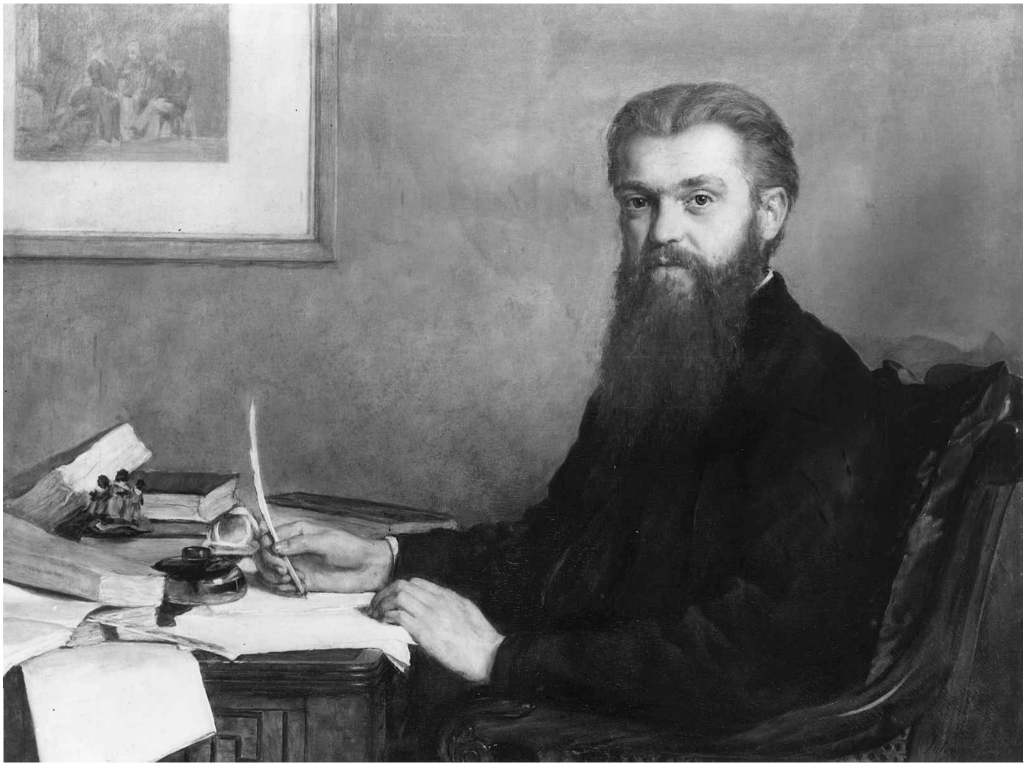

Figure 3. A portrait of William Kingdon Clifford, FRS (1845–1879), mathematician and philosopher, by the Hon. John Collier. (Royal Society Library and Archives.)

order of vectors in the outer product we change its sign:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

We are more used to dealing with a product that is commutative, i.e. multiplication between two numbers, $2 \times 5 = 5 \times 2 = 10$, but it turns out to be extremely useful in many areas of physics, maths and engineering to have a product that does not necessarily commute. By contrast, the *inner product* between two vectors, $\mathbf{a}$ and $\mathbf{b}$, written as $\mathbf{a} \cdot \mathbf{b}$ (this produces a scalar whose magnitude is $ab\cos\theta$, where θ is the angle between the vectors), is *commutative*, i.e.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

Grassmann, a German schoolteacher, was largely ignored during his lifetime, but since his death his work has stimulated the fashionable areas of *differential forms* and *Grassmann* (anticommuting) *variables*. The latter are fundamental to the foundation of much of modern supersymmetry and superstring theory.

The next crucial stage of the story occurs in 1878 with the work of the English mathematician, William Kingdon Clifford (Clifford 1878; see figure 3). Clifford was one of the few mathematicians who had read and understood Grassmann's work, and in an attempt to unite the algebras of Hamilton and Grassmann into a single structure, he introduced his own *geometric algebra*. In this algebra we have a single *geometric product* formed by uniting the inner and outer products; this is associative like Grassmann's product but also *invertible*, like products in Hamilton's algebra. In Clifford's geometric algebra, an equation of the type $\mathbf{ab} = C$ has the solution $\mathbf{b} = \mathbf{a}^{-1}C$, where $\mathbf{a}^{-1}$ exists and is known as the *inverse* of $\mathbf{a}$. Neither the inner nor the outer product possess this invertibility on their own. Much of the power of geometric algebra lies in this property of invertibility.

Clifford's algebra combined all the advantages of quaternions with those of vector geometry, so geometric algebra should then have gone forward as the main system for mathematical physics. However, two events conspired against this. The first was Clifford's untimely death at the age of just 34, and the second was *Gibbs's* introduction of his *vector calculus*. Vector calculus was well suited to the theory of electromagnetism as it stood at the end of the 19th century; this, and Gibbs's considerable reputation, meant that his system eclipsed the work of Clifford and Grassmann. It is ironic that Gibbs himself seems to have been convinced that Grassmann's approach to multiple algebras was the correct one.† With the arrival of special relativity, physicists realized that they were in need of a system capable of handling four-dimensional space, but, by this time, the crucial insights of Grassmann and Clifford had long been lost in the papers of the late 19th century.

In the 1920s, Clifford algebra resurfaced as the algebra underlying *quantum spin*. In particular, the algebra of *Pauli* and *Dirac* spin matrices became indispensable in quantum theory. However, they were treated just as algebras: the *geometrical* meaning had been lost. Accordingly, we will employ the term 'Clifford algebras' when the use is solely in formal algebra. When applied in its proper, geometric setting, we use Clifford's own name of *geometric algebra*. This is also a concession to Grassmann, who was actually the first to write down a geometric (Clifford) product!

The situation remained largely unchanged until the 1960s, when David Hestenes began to recover the geometric meaning behind the Pauli and Dirac algebras (Hestenes 1966). Although his original motivation was to gain some insight into the nature of quantum mechanics, he very soon realized that, properly applied, Clifford's system was nothing less than a universal language for mathematics, physics and engineering. Again, this remarkable work was largely ignored for around 20 years, but today interest in Hestenes's system (Hestenes & Sobczyk 1984; Hestenes 1986) is gathering momentum. There are now many groups around the world working on applying geometric algebra to topics as diverse as black holes and cosmology, quantum tunnelling and quantum field theory, beam dynamics and buckling, computer vision and robotics, protein folding, neural networks, and computer-aided design (Sommer 2000; Doran *et al*. 1996; Baylis 1996; Lasenby *et al*. 1998). Exactly the same algebraic system is used throughout, making it possible for people to make contributions across a number of these fields simultaneously.

3. Geometric algebra, a brief outline

In our geometric algebra we start out with *scalars*, i.e. ordinary numbers that have a magnitude but no associated orientation, and vectors, i.e. directed line segments with both magnitude and orientation/direction. Let us now take these vectors and look a little more closely at the geometry behind Grassmann's outer product. The outer product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a} \wedge \mathbf{b}$ and is a *new* quantity

† In the chapter on *multiple algebras* in Gibbs (1906), Gibbs goes to great length in his discussion of the merits of the Grassmannian system.

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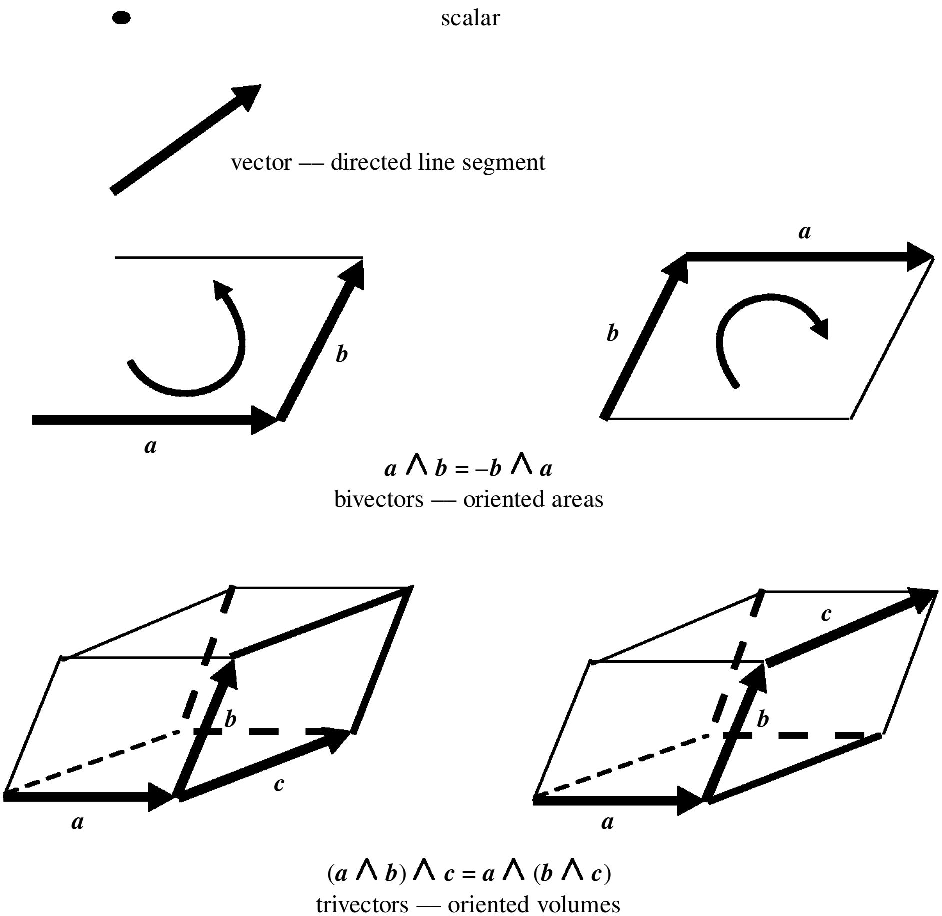

Figure 4. Vectors, bivectors and trivectors shown as oriented geometric objects.

called a *bivector*. The bivector $\mathbf{a} \wedge \mathbf{b}$ is the *directed area* swept out by the two vectors $\mathbf{a}$ and $\mathbf{b}$; thus the outer product of two vectors is a new mathematical entity that encodes the notion of an oriented plane. If we sweep $\mathbf{b}$ out along $\mathbf{a}$, we obtain the same bivector but with the opposite sign (orientation). Now, by extending this idea, we see that the outer product between three vectors, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$, is obtained by sweeping the bivector $\mathbf{a} \wedge \mathbf{b}$ out along $\mathbf{c}$, thus giving an oriented volume or trivector. If we sweep $\mathbf{a}$ across the area represented by the bivector $\mathbf{b} \wedge \mathbf{c}$, we get the same trivector (it can be shown that it has the same 'orientation'); this fact expresses the associativity of the outer product. Figure 4 summarizes these ideas of the basic elements of the algebra as geometric objects. In an n-dimensional space, we can have *n-vectors*, which are simply *oriented n-volumes*; thus we see that the outer product is easily generalizable to higher dimensions, unlike the Gibbs's vector product, which is restricted to three dimensions.

The crucial step in developing geometric algebra now comes with the introduction of the *geometric product*. We already know what $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ are: the geometric product unites these in the single product $\mathbf{ab}$,

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.$$

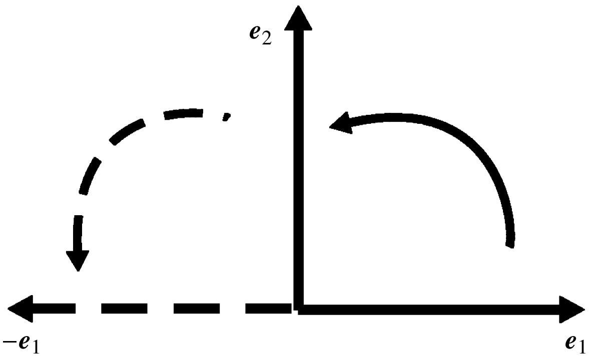

Figure 5. Multiplication on the right by the bivector $\mathbf{e}_1\mathbf{e}_2$ rotates 90° anticlockwise.

This step of summing two *different* objects is not a totally foreign act; in fact, we have long been doing a similar thing when carrying out operations with complex numbers. It turns out that many quantities in physics can be expressed very concisely and efficiently in terms of *multivectors* (linear combinations of n-vectors, e.g. a scalar plus a bivector, etc.); indeed, this combining of objects of different types appears to occur at a deep level in physical theory.

(*a*) *Geometric algebra in two dimensions*

In two dimensions (a plane), any point can be reached by taking different linear combinations of two vectors with different directions; we say the space is then *spanned* by these two *basis* vectors. Now let these two vectors be *orthonormal*, i.e. of unit length and perpendicular to each other, and call them $\mathbf{e}_1$ and $\mathbf{e}_2$. They satisfy

$$\mathbf{e}_1{}^2 = \mathbf{e}_2{}^2 = 1, \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0,$$

which are the equations that encode these properties. The only other element in our two-dimensional geometric algebra is the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$; this is the highest grade element in the algebra (often called the *pseudoscalar*).

Let us now look at the properties of this bivector. The first thing to note is that

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1,$$

i.e. the geometric product is a pure bivector because the perpendicularity of the vectors guarantees that $\mathbf{e}_1 \cdot \mathbf{e}_2$ vanishes. Now let us square this bivector:

$$(\mathbf{e}_1\mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -(\mathbf{e}_1)^2(\mathbf{e}_2)^2 = -1.$$

Note that we have a real geometric quantity that squares to -1! It is therefore tempting to relate this quantity with the unit imaginary of the complex number system (a complex number takes the form $x + \mathrm{i}y$ where the i is known as the *unit imaginary* and has the property that $\mathrm{i}^2 = -1$). Thus, in two dimensions, geometric algebra reproduces the properties of the complex numbers but uses only geometric objects. In fact, going to geometric algebras of higher dimensions, we begin to see that there are *many* objects that square to -1, and that we can use them all in their correct geometric setting.

Let us now see what happens when the bivector $\mathbf{e}_1\mathbf{e}_2$ multiplies vectors from the left and right. Multiplying $\mathbf{e}_1$ and $\mathbf{e}_2$ from the left gives

$$(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1 = -\mathbf{e}_2,$$
$$(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_1.$$

We therefore see that left multiplication by the bivector rotates vectors 90° *clockwise*. Similarly, if we multiply on the right we rotate 90° *anticlockwise* (see figure 5):

$$\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_2, \qquad \mathbf{e}_2(\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_1.$$

4. Rotations

From the properties of the bivector $\mathbf{e}_1\mathbf{e}_2$, it is then very easy to show that a rotation of a vector $\mathbf{a}$ through an angle θ to a vector $\mathbf{a}'$ is achieved by the equation

$$\mathbf{a}' = R\mathbf{a}\tilde{R},$$

where R is a quantity we shall call a *rotor* and is made up of a scalar plus a bivector,

$$R = \cos\tfrac{1}{2}\theta - \mathbf{e}_1\mathbf{e}_2\sin\tfrac{1}{2}\theta,$$

and $\tilde{R}$ is the same expression but with a '+'. This may at first seem like a rather cumbersome expression to deal with in order to carry out a simple two-dimensional rotation; however, it turns out that it is generalizable to higher dimensions and therefore has enormous power.

The above equation, $\mathbf{a}' = R\mathbf{a}\tilde{R}$, is, in fact, the formula that is used to rotate a vector in any dimension; if we go to three dimensions, the rotor R will rotate by an angle θ in the plane described by a bivector. Therefore, all we need do is replace the bivector $\mathbf{e}_1\mathbf{e}_2$ with the bivector that defines the plane of rotation (see figure 6). And that is all there is to it; using this very simple expression we find that we can not only rotate vectors, but also bivectors and higher-grade quantities. To carry out rotations in three dimensions in a manner that extended the concepts we understood in two dimensions was a problem Hamilton struggled with for many years, before finally producing, as his solution, the *quaternions*. In fact, the elements of Hamilton's quaternion algebra are nothing other than elementary bivectors (planes).

Having this very simple idea of a *rotor* that performs rotations, we can give amazingly simple geometric interpretations of many otherwise complicated fields; some examples are given below.

5. Special relativity

Special relativity was introduced in 1905 and heralded the beginning of a new era in physics; the departure from the purely classical regime of Newtonian physics. In special relativity (SR), we deal with a four-dimensional space; the three dimensions of ordinary Euclidean space, and time. Suppose we have a stationary observer with whom we can associate coordinates of space and time, this observer will observe events from his *space-time* position. Now suppose that we have another observer travelling at a velocity $\mathbf{v}$; he too will observe events from his continuously changing space-time position. The problem of how the two observers perceive different events

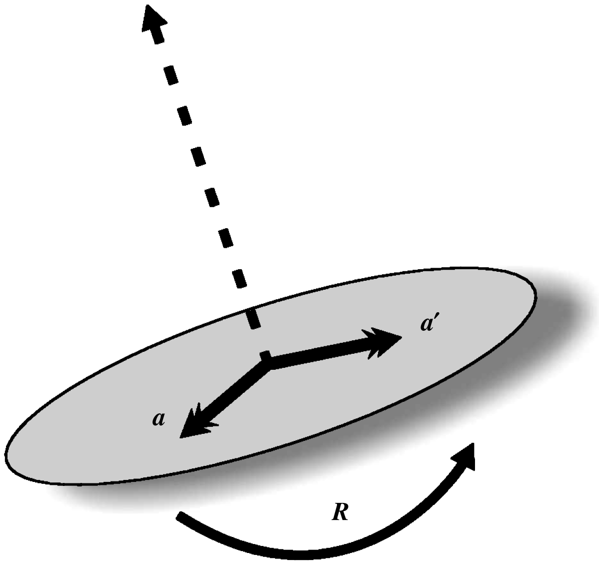

Figure 6. The rotor R taking the vector $\mathbf{a}$ to the vector $\mathbf{a}'$. Note that the concept of the perpendicular vector is no longer needed; it is the bivector or plane of rotation that is important.

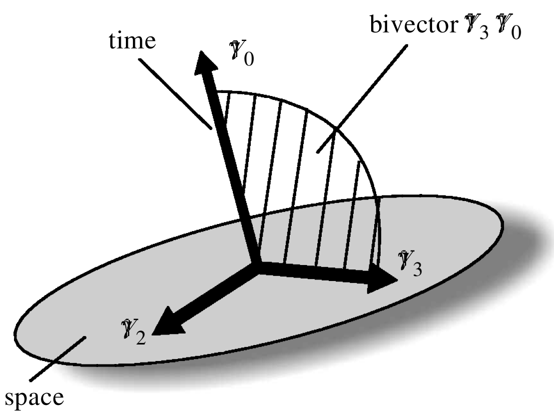

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Figure 7. Illustration of the four-dimensional space-time axes. One of the time-space bivectors is shown; as before it defines a plane in our space and can therefore be used in rotating the axes.

is relatively easy when the speed, $|\mathbf{v}|$, is small. But, when $|\mathbf{v}|$ approaches the speed of light, c, and we add in the fact that c must be constant *in any frame*, the mathematics is no longer so straightforward. Conventionally, one can derive a coordinate transformation between the frames of the two observers, and to move between these

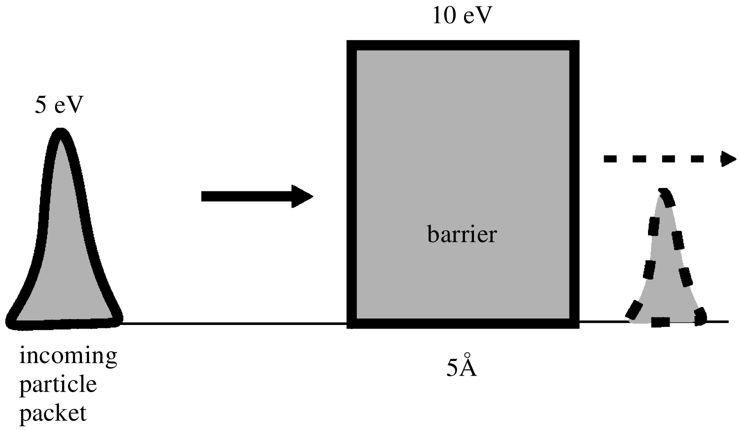

Figure 8. Particle packet incident on a barrier of higher energy than itself.

two frames we apply a matrix transformation known as a *Lorentz boost*. Geometric algebra provides us with a beautifully simple way of dealing with special relativistic transformations using nothing other than the formula for rotations discussed above, namely $\mathbf{a}' = R\mathbf{a}\tilde{R}$ (Hestenes 1966; Gull *et al.* 1993). Our space now has four dimensions and our basis vectors are the three space directions and one time direction; let us call these basis vectors γ_0, γ_1, γ_2 and γ_3. Because we have four dimensions we have six bivectors (the three spatial bivectors plus the bivectors made up of space-time 'planes'). The Lorentz boost turns out to be simply a rotor R, which takes the time axis to a different position in four dimensions: $R\gamma_0\tilde{R}$ (see figure 7). So, in an elegant coordinate-free way we are able to give the transformations of SR an intuitive geometric meaning. All the usual results of SR follow very quickly from this starting point. For example, the complicated formulae for the transformation of the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields under a Lorentz boost are replaced by the (much simpler!) result

$$\mathbf{E}' + I\mathbf{B}' = R(\mathbf{E} + I\mathbf{B})\tilde{R},$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3$ is the pseudoscalar of four-dimensional space (a 4-volume) and primes denote transformed quantities.

6. Quantum mechanics

In non-relativistic quantum mechanics, there are important quantities known as *Pauli spinors*; using these spinors we are able to write down an equation (the *Pauli equation*) that governs the behaviour of a quantum mechanical state in some external field. The equation involves quantities called *spin operators*, which are conventionally seen as completely different entities to the *states*. Using the three-dimensional geometric algebra we are able to write down the equivalent to the Pauli equation in which the operators and states are all real-space multivectors; indeed, the spinors become rotors of the type we have discussed earlier.

Now, the extension to *relativistic quantum mechanics* is easy. Conventionally, this is described by the Dirac algebra, where the *Dirac equation* again tells us about the state of the particle in an external field. This time we use the four-dimensional space-time geometric algebra, and once again the *wavefunction* in conventional quantum mechanics becomes an instruction to rotate a basis set of axes and align them in certain directions: analogous to the theory of rigid-body mechanics! The simplicity of this approach has some interesting consequences. The Dirac equation for some external potential A can be solved, and, by seeing where the time axis, γ_0, has been rotated to, we can plot streamlines (lines that give the direction of the particle's velocity at each point) of the particle's motion. We can illustrate the comparison with conventional theory by a simple example. Consider the case of an incident particle packet, of energy 5 eV, say, encountering a rectangular barrier potential of height 10 eV and finite width, 5 Å say (see figure 8). The theory of quantum mechanics enables us to predict that despite the seemingly impenetrable barrier, some of the packet indeed emerges the other side—an effect called *tunnelling*, which is of fundamental importance in many of today's semiconductor devices. However, when we ask the apparently obvious question of *how long does a tunnelling particle spend inside the barrier*, quantum theory provides us with a variety of answers:

(a) this cannot be discussed as *time* is not a Hermitian observable;

(b) the time is identically zero;

(c) the time taken is imaginary.

Why should quantum mechanics make such strange predictions? The main reason for the inability to deal with the path of the particle/packet within the barrier lies in the use of i, the uninterpreted scalar imaginary ($i^2 = -1$); conventionally, the momentum of the particle within the barrier is taken as a multiple of i and this leads to these rather unhelpful ideas of imaginary time.

However, the geometric algebra approach tells us that we should plot the streamlines representing the path of the particle within the barrier, and, hence, find how much time they really spend inside the barrier. Not too far into the next millennium it may be possible to compare the times given by this theory with times measured in actual experiments. Figure 9 shows the predicted streamlines of particles starting at different positions within the wave packet of energy 5 eV incident on a barrier of height 10 eV and width 5 Å, as depicted above. It can be seen that the particle streamlines *slow up* while in the barrier. This is in contrast with some recent discussions of superluminal velocities within such barriers, which have been inferred from the experimentally observed fact that particles tunnelling through a barrier reach a target *before* those travelling an equivalent distance in free space. This apparent contradiction is explained here by the fact that it is particles near the front of the wave packet, which already have a head start, that are transmitted and able to reach the target.†

7. Gravity

Electromagnetism is a *gauge theory*. A gauge theory occurs if we stipulate that global symmetries must also become local symmetries (in electromagnetism these symmetries are called *phase rotations*); the price one has to pay to achieve this is the introduction of *forces*. In geometric algebra, gravity can also be regarded as a gauge

† It is interesting to note that much of the currently fashionable area of *quantum cosmology* is based on the concepts of imaginary time.

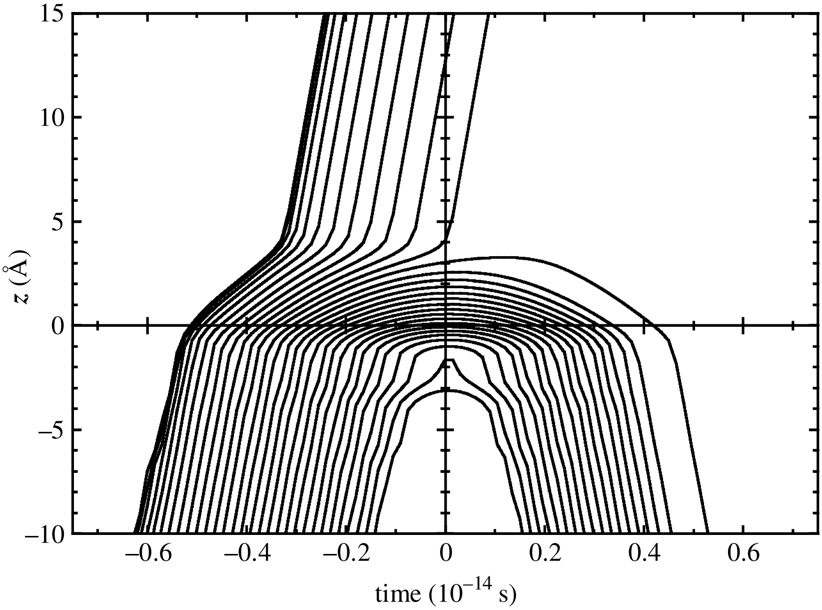

Figure 9. Streamlines of particles (energy 5 eV) incident on a barrier (energy 10 eV). z is the distance (in angstroms) in the direction of travel and the barrier is between $z = 0$ and $z = 5$. Particles start out at different positions within the wave packet, illustrated by the spread of lines along the $z = -10$ axis. Particles near the front of the wave packet are transmitted whereas those near the back are reflected.

theory, and here the symmetries are much easier to understand. Suppose we require that physics at all points of space-time is invariant under arbitrary local displacements and rotations (recall that by rotations in four dimensions we are referring to Lorentz boosts); the *gauge field* that results from such a requirement is the gravitational field. A consequence of this theory is the huge simplification of being able to discuss gravity entirely in a *flat space-time* background (Lasenby *et al*. 1998). There is no need for the complex notions of curved space-time that we are all used to associating with Einstein's theory of general relativity (GR). This is where the GA approach differs from past gauge-theoretic approaches to gravity—these past theories have still retained the ideas of a curved space-time background. *Locally*, the GA gauge theory of gravity reproduces all the results of general relativity, but *globally*, the two theories will differ when issues of topology are at hand. For example, whenever there is discussion of singularities or horizons (as with black holes), the GA theory can give different predictions to conventional GR. Some improved methods for *solution*, working entirely with physical quantities, have also been found in GA.

The GA gauge theory of gravity deals with extreme fields (i.e. fields in which singularities occur) in a different manner to GR. These singularities are treated simply in a manner analogous to that employed in electromagnetism (using *integral theorems*). The interaction with quantum fields is also different and suggests an alternative route to a quantum theory of gravity. In this context, it is also interesting to note that many of the other fashionable attempts at uniting gravity and quantum theory (twistors, supergravity, superstrings) also sit naturally within the GA framework.

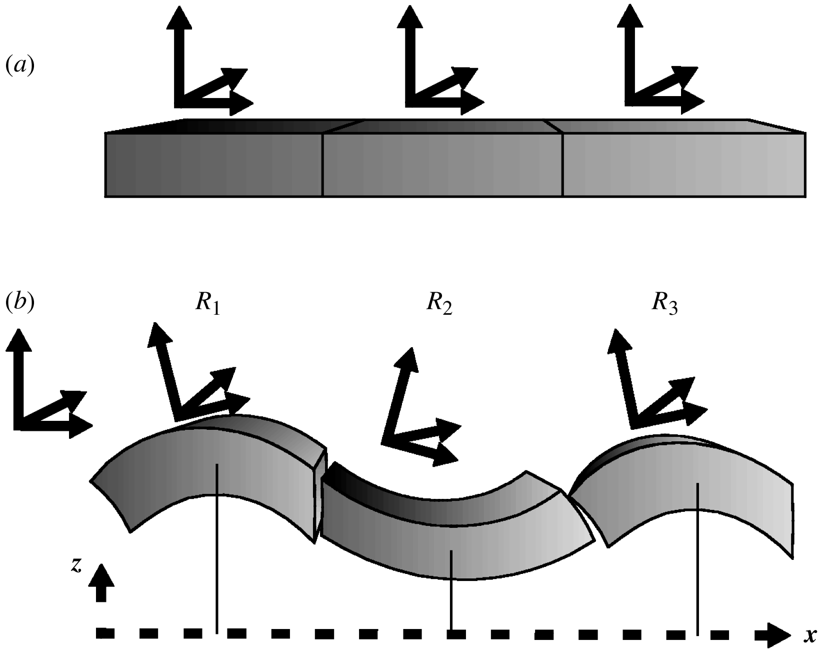

Figure 10. Model of a beam split into very small segments; the deformation is described by the position and orientation of each segment.

8. Rods, shells and buckling beams

It is not only in the areas of fundamental physics that geometric algebra is a useful tool. The concept of a frame of reference that varies in either space or time (or both) is at the heart of much work that tries to understand deforming bodies. Let us take, as a simple example, a beam of uniform cross-section that is subject to some loading along its length; the properties of the beam and the loading will determine how the beam deforms. Mathematically, we can describe the deformation by splitting up the beam into very small segments and attaching a frame (three mutually perpendicular axes) to the centre of mass of each segment. Initially, under no loading and no torsion, we expect the origin, O, of each frame to be along the centreline of the beam and that each frame is aligned so that the x-axis points along the length of the beam and the z-axis vertically upwards. Now, as the beam deforms, we can describe its position at a given time by specifying the position of the origin and the orientation of the frame for each segment.

Suppose we have a fixed frame at one end of the beam, the frame at segment i will then be related to this fixed frame by some rotor, R_i. Thus, as we move along the beam, the orientations are described by a rotor that varies with distance x (see figure 10). For a given loading and specified boundary conditions one might want to solve for the rotors to give information on the buckling properties of the beam. Conventionally, this task has been carried out using a variety of means to encode rotations; Euler angles, rotational parameters, direction cosines, rotation matrices, etc. The advantage of using rotors is twofold; firstly, they automatically have the correct number of degrees of freedom (three), unlike, for example, direction cosines (where we have nine parameters, only three of which are independent), and secondly, we can solve the full equations (without approximations) in an efficient manner.

One can take this idea of varying frames one stage further. Today, much of the research in modern structural mechanics has become the province of the mathematician. In order to deal with thin structures such as rods and shells, where, under deformation, the surface structure can be fairly complicated, people saw that areas of mathematics such as *differential geometry* and *differential topology* might provide useful tools. Indeed, much of the finite-element code used today in standard structural engineering packages is written from algorithms based on this mathematics. The outcome is, however, that many of the engineers can no longer understand the working of such packages, and must take for granted that what they are using is correct. On the other hand, using geometric algebra, the problem again reduces to having rotors that may vary in time and/or space across any given surface; the mathematics is no harder than one would use to solve simple mechanics problems (McRobie & Lasenby 1999). The internal finite-element code thus becomes accessible to engineers and modifications are possible.

9. Computer vision and motion analysis

Computer vision is essentially the art of reconstructing or inferring things about the real three-dimensional world from views of the scene taken by one or several cameras. The positions and orientations of the cameras may or may not be known and the internal parameters of the cameras (which determine how the images we see differ from those that would result from a perfect projection onto an image plane) may also be unknown. From this rather simplified description, one can see that a significant amount of three-dimensional geometry will be involved. In fact, since the mid-1980s much of computer vision has been written in the language of *projective geometry* (Faugeras 1993). In classical projective geometry, we define a three-dimensional space whose points correspond to lines through some origin (specified point) in a four-dimensional space. Using such a system, the algebra of incidence (intersections of lines, planes, etc.) is extremely elegant, and, moreover, transformations that appear complicated in three dimensions (e.g. projection of points, lines, etc., down onto a given plane) now become simple. In recent years, people have started to use an algebra called the *Grassmann–Cayley algebra* for projective geometry calculations and manipulations; this is effectively Grassmann's exterior algebra as it restricts itself to using only the outer product. Geometric algebra contains the exterior algebra as a subset and is, therefore, an ideal language for expressing all the ideas of projective geometry (Hestenes & Ziegler 1991; Lasenby & Bayro-Corrochano 1997). However, GA also has the notion of an inner product, which often allows us to do things that would be very difficult with only an outer product.

To illustrate another way in which geometric algebra can be used in computer vision, let us look at a problem that occurs in motion analysis (the reconstruction of the three-dimensional motion of an object from the image coordinates of matched points in several camera views), in scene reconstruction and image registration (mosaicking a number of different, overlapping images when limited information is available). Suppose we have a number of cameras observing an object, we suppose also, for convenience, that markers are placed on the object so that these points

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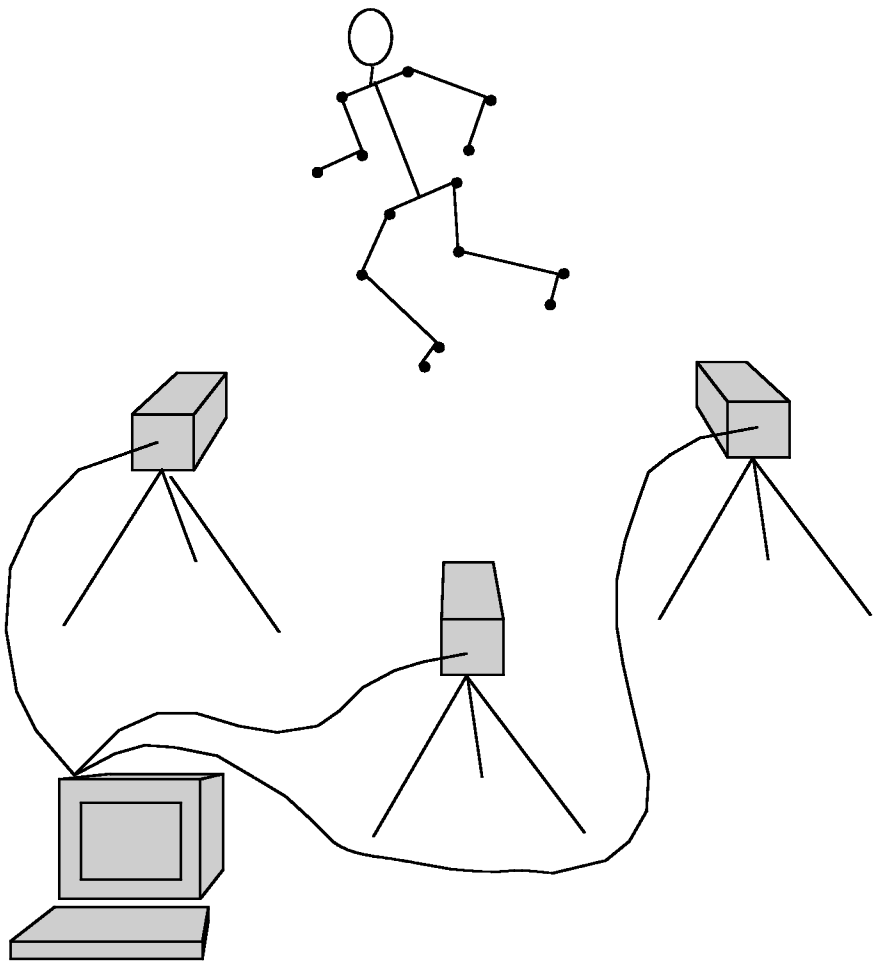

Figure 11. Schematic showing a marked object observed by a system of cameras feeding data back to the processor.

can be easily extracted from the images. Figure 11 shows a sketch of a three-camera system.

Now, if we observe a scene with, say, M cameras, we will find that in each pair of cameras there is a subset of the total number of markers that are visible. The first problem is to find, using these M images, the best estimates of the relative positions and orientations of each camera. Once we know the positions of the cameras we would like to *triangulate* in order to find the three-dimensional coordinates of other world points visible in a number of images; these problems are not too difficult for exactly known image points but become much harder if these points are noisy. There do of course exist conventional techniques for solving these problems, indeed, photogrammetrists have been doing precisely this for many years. However, the solutions generally involve large optimizations, which are often unstable. This is where geometric algebra can help. Using GA, it is possible to solve both the calibration and triangulation problems in a way that takes into account all the data from each camera simultaneously. The optimizations involved in the solutions are able to use both first and second analytic (as opposed to numerical) derivatives† of all quantities to be estimated in a consistent way. Conventionally, it is much harder to take derivatives of quantities representing rotations. Using GA in this way, it is possible to produce accurate solutions while reducing the computational load, thus making it useful in applications that require many such optimizations.

10. Conclusions

We have attempted to give a brief introduction to the mathematical system we refer to as *geometric algebra* and to illustrate its usefulness in a variety of fields. While we have discussed a range of topics from quantum mechanics to buckling beams, there are many persuasive examples of the use of GA in physics and engineering that we have not discussed. These include electromagnetics, polarization, geometric modelling and linear algebra. The modern tools of mathematics, of which most of us are familiar with but a few, are varied and complex. In one lifetime of research we can hope only to master a very few areas. However, if most of physics and mathematics were to use the *same* language, the situation would perhaps be different. We hope that we have shown in this paper that geometric algebra is a candidate for such a unified language.

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† A derivative is simply the rate of change of a quantity; for example, speed and acceleration are the first and second derivatives of distance with respect to time.

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